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AN EXTENSION OF HENCKY'S THEOREM†

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An extension of Hencky's theorem is achieved by using Coulomb's plasticity condition as its physical basis. This gives relations between the mean normal stresses and the principal directions at the nodal points of the mesh of characteristics, which reduce to Hencky's theorem in the case of ideally plastic media. Copyright © 1996 Elsevier Science Ltd.

The plastic properties of soils can be described by Coulomb's law

$$|\tau_n| = \bar{\sigma}_n \operatorname{tg}\rho \tag{1}$$

where τ_n is the shear stress in slip areas, $\bar{\sigma}_n = H + \sigma_n$ is the reduced normal stress, $H = c/\operatorname{tg}\rho$ is the hydrostatic pressure (connectedness), which is equivalent to the cohesion C, ρ is the angle of internal friction of the soil under conditions of plane deformation and σ_n is the normal stress in the slip areas.

Under conditions of plane strain, the components of the stress which satisfy (1) are expressed by the relations

$$\bar{\sigma}_x, \bar{\sigma}_y = \bar{\sigma}(1 \pm \sin\rho\cos 2\chi), \quad \tau_{xy} = \bar{\sigma}\sin\rho\sin 2\chi \tag{2}$$

where $\bar{\sigma} = H + \sigma$ is the reduced mean normal stress σ, $\bar{\sigma}_x$, $\bar{\sigma}_y$ are the components of the reduced normal stress, τ_{xy} is a component of the shear stress and χ is the angle of inclination of the principal direction to the x axis. The introduction of expressions (2) into the equilibrium equations for a ponderable medium (γ is the bulk (specific) weight of the medium) gives a basic system of Kötter equations of the hyperbolic type with a system of characteristic equations [1]

$$dy = dx \operatorname{tg}(\chi \mp \varepsilon) \tag{3}$$

where $\varepsilon = \pi/4 - \rho/2$ and a system of relations along the characteristics

$$d\sigma \mp 2\bar{\sigma}\operatorname{tg}\rho\, d\chi = \gamma(dy \mp dx\operatorname{tg}\rho) \tag{4}$$

in which the upper signs refer to the first family of characteristics (α) and the lower signs refer to the second family of characteristics (β).

Equations (3) and (4) in difference form are commonly used for the numerical solution of boundary-value problems by Masso's method. The proposed solution of the problem of the extension of Hencky's theorem which has been formulated required the use of a difference form of Eqs (4) for a weightless medium ($\gamma = 0$). They take the form

$$\bar{\sigma} - \bar{\sigma}_a = 2\bar{\sigma}_a(\chi - \chi_a)\operatorname{tg}\rho \tag{5}$$

$$\bar{\sigma} - \bar{\sigma}_b = -2\bar{\sigma}_b(\chi - \chi_b)\operatorname{tg}\rho \tag{6}$$

where the subscripts a and b indicate that the corresponding quantities belong to the nodal points of a mesh of characteristics (see Fig. 1); Eq. (5) refers to the first family and Eq. (6) to the second family.

In accordance with the scheme for an element of a medium which is bounded by two pairs of characteristics α and β (Fig. 1), we determine, by solving the initial characteristic problem, the values of χ and $\bar{\sigma}$ at the lower point of the "rhombus" by eliminating them from Eqs (5) and (6). We finally obtain

$$\chi = \frac{\bar{\sigma}_b - \bar{\sigma}_a + 2(\bar{\sigma}_a\chi_a + \bar{\sigma}_b\chi_b)\operatorname{tg}\rho}{2(\bar{\sigma}_a + \bar{\sigma}_b)\operatorname{tg}\rho} \tag{7}$$

$$\bar{\sigma} = 2\frac{\bar{\sigma}_a\bar{\sigma}_b[1 + (\chi_b - \chi_a)\operatorname{tg}\rho]}{\bar{\sigma}_a + \bar{\sigma}_b}$$

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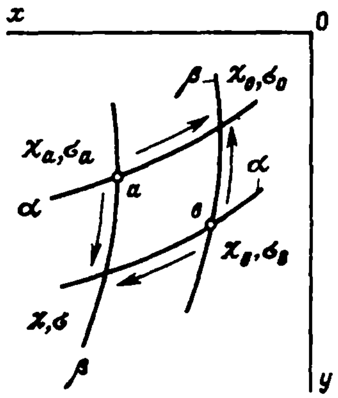

Fig. 1.

To determine χ_0 and $\bar{\sigma}_0$ at the upper point of the "rhombus", it is sufficient to interchange $\bar{\sigma}_a$, χ_a and $\bar{\sigma}_b$, χ_b in these expressions.

On adding the resulting expressions, we obtain

$$\chi+\chi_0=2\frac{\bar{\sigma}_a\chi_a+\bar{\sigma}_b\chi_b}{\bar{\sigma}_a+\bar{\sigma}_b},\quad \bar{\sigma}+\bar{\sigma}_0=4\frac{\bar{\sigma}_a\bar{\sigma}_b}{\bar{\sigma}_a+\bar{\sigma}_b} \tag{8}$$

which is the extended Hencky's theorem for a soil medium in plastic equilibrium.

Substituting the values of $\bar{\sigma}=H+\sigma$ and $H=c/\text{tg}\,\rho$ into relation (8) when $\text{tg}\,\rho=0$, that is, for an ideally plastic medium, we obtain the well-known Hencky relations

$$\chi+\chi_0=\chi_a+\chi_b,\quad \sigma+\sigma_0=\sigma_a+\sigma_b \tag{9}$$

(equality of the "diagonal" sums of the mesh points of the principal directions and the sums of the mean normal stresses at the mesh points of the "square" formed by the characteristics).

Relations (8) are convenient for the numerical solution of problems involving a weightless medium as well as for a ponderable soil medium when the first relation of (8) can be used as a first approximation when determining the coordinates of the nodal points of the mesh of the characteristics using the formulae

$$x=\frac{x_a\,\text{tg}(\bar{\chi}_a-\varepsilon)-x_b\,\text{tg}(\bar{\chi}_b+\varepsilon)-y_a+y_b}{\text{tg}(\bar{\chi}_a-\varepsilon)-\text{tg}(\bar{\chi}_b+\varepsilon)} \tag{10}$$

$$y=(x-x_a)\,\text{tg}(\bar{\chi}_a-\varepsilon)+y_a\quad \text{or}\quad y=(x-x_b)\,\text{tg}(\bar{\chi}_b-\varepsilon)+y_b$$

which are obtained by solving Eqs (3) in difference form, where $\bar{\chi}_a$ and $\bar{\chi}_b$ are assumed to be equal to $\bar{\chi}_a=(\chi_a+\chi)/2$ and $\bar{\chi}_b=(\chi_b+\chi)/2$ in which χ is the first approximation found using the above-mentioned relation. The values of $\bar{\sigma}$ and the first approximation of χ can be found using the recurrence formulae

$$\chi=\Sigma_b-\Sigma_a,\quad \bar{\sigma}=2(\bar{\sigma}_b\Sigma_a+\bar{\sigma}_a\Sigma_b)\,\text{tg}\rho \tag{11}$$

in which

$$\Sigma_a=\frac{\gamma[(y-y_a)-(x-x_a)\,\text{tg}\rho]+\bar{\sigma}_a(1-2\chi_a\,\text{tg}\rho)}{2(\bar{\sigma}_a+\bar{\sigma}_b)\,\text{tg}\rho} \tag{12}$$

and Σ_b differs from Σ_a in that the subscript a in the numerator is replaced by b and the signs of tg ρ are replaced by the opposite signs. These formulae are required to solve the initial characteristic problem. The coordinates of the mesh points (10) are then refined using the second approximation of χ, while $\bar{\sigma}$ is refined by means of a corresponding repeated calculation with which the computational process can be concluded, where the values of χ_a and χ_b are not corrected.

REFERENCE

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